Lecture 5: Vector Spaces

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4.1 An example different from $\mathbb{R}^n$

$E_1: x - x - 2 = -1$
$E_2: 2x - y + 2 = 1$
$E_3: -x + 2y + 4z = 4$

using these equations, we can construct new ones

$E_2 - 2E_1:\ y + 3z = 3$
$E_1 + E_3:\ y + 3z = 3$
So: $E_2 - 2E_1 = E_1 + E_3$

We can check that our properties V1-V4, S1-S4 (from lecture 3) are satisfied.

4.2 Definition

We need a set V of vectors, a rule "+" for adding two vectors, and a rule "·" for multiplying a scalar and a vector such that the following axioms hold:

Closure

(1) The sum of any 2 vectors is again a vector.
$\forall\ u, v \in V:\ u + v \in V$
(2) Any scalar multiple of any vector is again a vector.
$\forall\ c \in \mathbb{R}, u \in V:\ c * u \in V$

Existence

(1) There is a zero vector.
$\exists 0 \in V\ \ \forall u \in V\ : 0 + u = u$
(2) Every vector has a negative.
$\forall u \in V, \exists\ -\mathrm{u} \in V: u + (-u) = 0$

Arithmetic Properties

For any $u, v, w \in \mathbb{R}$ and $c, d \in \mathbb{R}$:
(1) "Commutativity" $u + v = v + u$
(2) "Associativity" $u + (v + w) = (u + v) + w$
(3) "Distributivity" $c(u + v) = cu + cv$
(4) "Associativity" $c(du) = (cd)u$
(5) "Neutrality" $1u = u$

Any set V with 2 operations ("+" and "·") that satisfies the above axioms is called a **vector space**.
We've already seen the vector space:
$\mathbb{R}^n (n \in N)$
$\{equations\ in\ x, y, z\}$ with their addition and scalar multiplication.

4.3 Coming back to 4.1

$\mathcal{E} = \left\{ k_1 E_1 + k_2 E_2 + k_3\ E_3 \middle| k \in \mathbb{R} \right\}$
subset $\subseteq \{equations\ in\ x, y, z\}$
$\mathcal{E}$ is the set of all equations obtainable from E_1, E_2, E_3, and is also a **vector space**.

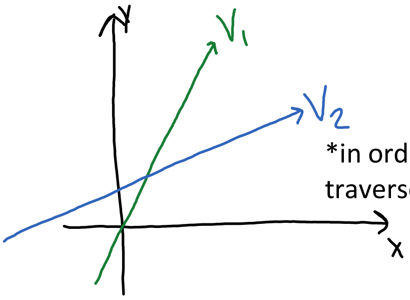
We can now formulate:

- can we solve E_1, E_2, E_3 for x?
- are there $k_1, k_2, k_3, x_0 \in \mathbb{R}$ such that "x=x_0"is obtained from $k_1 E_1 + k_2 E_2 + k_3 E_3$?
- is there $x_0 \in \mathbb{R}$ such that "x=x_0 is a linear combination of E_1, E_2, E_3?
- is there an $x_0 \in \mathbb{R}$ such tthat"x=x_0" $\in \mathcal{E}$? ← similar to is some point on some plane

4.4 Further examples

a) $V = \{0\}$ with $0 + 0 = 0$ and $c * 0 = 0$ for all $c \in \mathbb{R}$

b) $V_1 = \{(x, 2x) | x\epsilon\mathbb{R}\}$ inherited from $\mathbb{R}^2$ (standard operations)
$V_2 = \{(x, x+2) | x\epsilon\mathbb{R}\}$ with standard operation is **not** a vector space

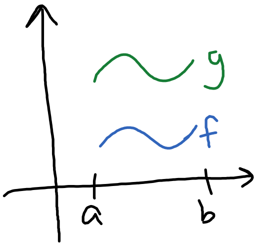


*in order to be a vector space, the line must traverse the origin

c) Def^n: A matrix is a table of numbers. If it has m rows and n columns we say it has size $n * m.$
eg. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is of size $2 * 3$
$M_{mn}(\mathbb{R}) = \{\text{all matrices of size } m * n \; with\ entries\ from\ \mathbb{R}\}$

eg. of a matrix vector space:
$V = M_{22}(\mathbb{R})$ **with** componentwise addition and componentwise scalar multiplication
<u>Use the subspace test:</u>
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 12 & 14 \end{bmatrix}$
$2 * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$
etc... (check all axioms and you can determine that it is in fact a vector space)

d) Let $[a, b] = \{x\epsilon\mathbb{R} | \ a \le x \le b\}$
$V = F[a, b] = \{f: [a, b] \to \mathbb{R}\}$



with $(f + g)(x) = f(x) + g(x)$
and $(c * f)(x) = c * f(x)$

e) $V = \mathrm{F}(\mathbb{R}) = \{f | f: \mathbb{R} \to \mathbb{R}\}$